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Finite-Dimensional Quantum Mechanics of a Particle

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This paper analyzes the possible implications of interpreting the finite-dimensional representations of canonically conjugate quantum mechanical position, and momentum operators of a particle consistent with Weyl's form of Heisenberg's commutation relation as the actual position, and momentum operators of the particle when it is confined to move within a finite spatial domain, and regarding the application of current quantum mechanical formalism based on Heisenberg's relation to such a situation as an asymptotic approximation. In the resulting quantum mechanical formalism the discrete and finite position and momentum spectra of a particle depend on its rest mass and the spatial domain of confinement. Such a "finite-dimensional quantum mechanics" may be very suitable for describing the physics of particles confined to move within very small regions of space.

1. INTRODUCTION

As is well known, though Heisenberg's quantum commutation relation

$$[\hat{q}, \hat{p}] = i\hbar \tag{1}$$

for position operator $\hat{q}$ and conjugate momentum operator $\hat{p}$ does not admit finite-dimensional representations, the equivalent Weyl's form of (1),

$$\hat{U}_\alpha \hat{V}_\beta = \exp(i\alpha\beta/\hbar)\hat{V}_\beta \hat{U}_\alpha \tag{2}$$

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with

$$\hat{U}_\alpha = \exp(i\alpha\hat{p}/\hbar) \tag{3}$$

$$\hat{V}_\beta = \exp(i\beta\hat{q}/\hbar) \tag{4}$$

has finite-dimensional realizations (Weyl, 1932). Based on this fact, Weyl (1932) analyzed quantum kinematics as an Abelian group of ray rotations of the system space and concluded as follows:

> We have thus found a very natural interpretation of quantum kinematics as described by the commutation relations. The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. The real elements of the algebra of this group are the physical quantities of the system; the representation of the abstract group by rotations of the system space associates with each such quantity a definite Hermitian form which "represents" it. If the group is continuous this procedure automatically leads to Heisenberg's formulation.... Our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group.... But the field of discrete groups offers many possibilities which we have not yet been able to realize in Nature; perhaps these holes will be filled by applications to nuclear physics. However, it seems more probable that the scheme of quantum kinematics will share the fate of the general scheme of quantum mechanics; to be submerged in the concrete physical laws of the only existing physical structure, the actual world.

Keeping in mind the above prophetic statement of Weyl, we shall analyze in this paper the possible implications of interpreting the finite-dimensional representations of $\hat{q}$ and $\hat{p}$ consistent with (2)–(4) as the actual position and momentum operators of a particle confined to move within a finite spatial domain, and regarding the application of current quantum mechanical formalism based on (1) to such a situation as only an asymptotic approximation.

Weyl (1932) and Schwinger (1960) studied the finite-dimensional version of (2) essentially as part of an intermediate step in a limiting process for the understanding of quantum mechanics based on (1). Many direct physical applications of the finite-dimensional form of (2) have been studied recently by Alladi Ramakrishnan and his collaborators (Alladi Ramakrishnan, 1972) as part of an extensive analysis of generalized Clifford algebras (see for instance, Alladi Ramakrishnan, 1971, 1972, for mathematical literature on generalized Clifford algebras). The present work stems from certain essential modifications of ideas on the possible existence of a quantum mechanics in discrete space, in view of Weyl's work on the finite-dimensional realization of (2), emphasized very much by one of us in recent years (see Santhanam and Tekumalla, 1976; Santhanam, 1977, 1978).

Models of discrete space-time have been discussed by physicists for a long time from many points of view (see for instance, Finkelstein, 1974; Lorente, 1974; Ginsburg, 1976; Dadic and Pisk, 1979; Stovicek and Tolar, 1979, for recent discussions of the subject and detailed references to earlier literature). The basic difference between our ideas presented here and the earlier ideas is that instead of an absolutely quantized space-time with universal minima of length and time or some kind of lattice structure common to all matter, the position spectrum of any particular particle is considered to depend on its rest mass, and the possible extent of its motion in space and time is considered to be an independent continuous parameter as usual. In this paper we shall limit ourselves to the detailed consideration of only one-dimensional space, and generalization to isotropic three-dimensional space should be straightforward when coordinate operators for different directions are assumed to commute as usual in the present quantum mechanics (see Snyder, 1947, for a different point of view).

2. FORMULATION OF THE FINITE-DIMENSIONAL QUANTUM MECHANICS

Throughout this paper we shall consider any $(2J+1)$-dimensional matrix to have its rows and columns labeled by integers from $-J$ to J as $\{-J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J\}$ and following the Dirac notation the nn'th matrix element of an operator or matrix M will be denoted by $\langle n|M|n'\rangle$. Then let N_J and Φ_J be $(2J+1)$-dimensional Hermitian matrices with

$$\langle n|N_J|n'\rangle = n\delta_{nn'} \tag{5}$$

$$\langle n|\Phi_J|n'\rangle = \frac{1}{(2J+1)} \sum_{s=-J}^{J} s \exp\left[\frac{i2\pi s(n-n')}{2J+1}\right]$$

$$= \begin{cases} 0 & \text{if } n=n' \\ \dfrac{i}{2}\csc\left[\dfrac{2\pi J(n-n')}{2J+1}\right] & \text{if } n\neq n' \end{cases} \tag{6}$$

The matrices N_J and Φ_J are related to each other as

$$\Phi_J = S_J N_J S_J^{-1} \tag{7}$$

where S_J is the unitary finite Fourier transform matrix defined by

$$\langle n|S_J|n'\rangle = \langle n|(S_J^{-1})^\dagger|n'\rangle = \frac{1}{(2J+1)^{1/2}} \exp\left(\frac{i2\pi nn'}{2J+1}\right) \tag{8}$$

This matrix S_J is equivalent to the $(2J+1)$-dimensional Sylvester matrix $\mathbb{S}_J$ with

$$\langle n|\mathbb{S}_J|n'\rangle = \frac{1}{(2J+1)^{1/2}}\exp\left[\frac{i2\pi(n+J)(n'+J)}{2J+1}\right] \tag{9}$$

Let us now define the Hermitian matrices,

$$Q_J = \epsilon_J N_J \tag{10}$$

$$P_J = \eta_J \Phi_J \tag{11}$$

where ϵ_J and η_J are real positive constants. Then the $(2J+1)$-dimensional unitary matrices,

$$U_{n\epsilon_J} = \exp(in\epsilon_J P_J/\hbar) \tag{12}$$

$$V_{l\eta_J} = \exp(il\eta_J Q_J/\hbar) \tag{13}$$

defined analogous to $\hat{U}_\alpha$ and $\hat{V}_\beta$, obey the relation

$$U_{n\epsilon_J} V_{l\eta_J} = \exp\{in\epsilon_J l\eta_J/\hbar\} V_{l\eta_J} U_{n\epsilon_J} \qquad \forall n, l=0, \pm 1, \pm 2, \ldots \tag{14}$$

exactly similar to (2) if

$$\frac{\epsilon_J \eta_J}{\hbar} = \frac{2\pi}{2J+1} \tag{15}$$

Further the matrices in (12) and (13) provide unique irreducible representation of (14) up to equivalence and constant multiplication factors (Weyl, 1932).

The finite Fourier transform matrix S_J of (8) has the following properties:

$$S_J Q_J S_J^{-1} = (\epsilon_J/\eta_J) P_J \tag{16}$$

$$S_J P_J S_J^{-1} = -(\eta_J/\epsilon_J) Q_J \tag{17}$$

$$S_J^2 Q_J S_J^{-2} = -Q_J \tag{18}$$

$$S_J^2 P_J S_J^{-2} = -P_J \tag{19}$$

$$\langle n|S_J^2|n'\rangle = \delta_{n,-n'} \tag{20}$$

$$S_J^4 = I_J \tag{21}$$

where I_J is the $(2J+1)$-dimensional identity matrix.

In our opinion a reasonable physical interpretation of the above set of relations (5)–(21) in the light of Weyl's remarks is that the quantum dynamics of a particle of rest mass m confined to move within a one-dimensional region of fixed length L might be based on the following principles:

(I) The position eigenvalues of the particle form a discrete and finite set $\{q_{Jn}\}$ given by

$$q_{Jn}=n\epsilon_J, \qquad n=-J,-J+1,\ldots,-1,0,1,\ldots,J-1,J. \tag{22}$$

and characterized by a "space quantum number" J such that

$$2J\epsilon_J \leqslant L < 2(J+1)\epsilon_{J+1} \tag{23}$$

when obviously there is no distinction between positive and negative directions with respect to the center of L taken as the origin of the coordinate system. In other words the quantum mechanical system space of the particle is $(2J+1)$-dimensional vector space with a unique value for the space quantum number J; corresponding particle operators are $(2J+1)$-dimensional matrices; and the basic position operator Q is given in position representation by

$$\langle n|Q|n'\rangle=\langle n|Q_J|n'\rangle=n\epsilon_J\delta_{nn'},$$
$$n,n'=-J,\ldots,-1,0,1,\ldots,J,$$
$$2J\epsilon_J \leqslant L < 2(J+1)\epsilon_{J+1} \tag{24}$$

Physically $2J\epsilon_J$ gives the dimension of the region of confinement of the particle.

(II) The momentum operator P conjugate to Q is given in position representation by

$$\langle n|P|n'\rangle=\langle n|P_J|n'\rangle=\frac{\eta_J}{2J+1}\sum_{s=-J}^{J} s\exp\left[\frac{i2\pi s(n-n')}{2J+1}\right]$$

$$=\begin{cases} 0 & \text{if } n=n' \\ \dfrac{i\eta_J}{2}\csc\left[\dfrac{2\pi J(n-n')}{2J+1}\right] & \text{if } n\neq n' \end{cases}$$

$$n,n'=-J,\ldots,-1,0,1,\ldots,J \tag{25}$$

and consequently the momentum eigenvalues of the particle also form a discrete and finite set $\{p_{Jn}\}$ such that

$$p_{Jn}=n\eta_J, \qquad n=-J,-J+1,\ldots,-1,0,1,\ldots,J-1,J \tag{26}$$

(III) The quantum of position ϵ_J and the quantum of momentum η_J are related to Planck's constant $\hbar$ as

$$\frac{\epsilon_J\eta_J}{\hbar}=\frac{2\pi}{2J+1} \tag{27}$$

(IV) Among the simplest dimensionless quantities that can be derived from the fundamental quantities, ϵ_J, η_J, m, $\hbar$, and the speed of light c, the most important are (i) $(\epsilon_J\eta_J/\hbar)$ due to its occurrence in the finite-dimensional version of Weyl's relation defined by (12)–(14) and (ii) $(\eta_J\hbar/\epsilon_J m^2c^2)$ due to the fact that the ratio (η_J/ϵ_J) occurs in the basic relations (16) and (17) connecting Q_J and P_J. Since the m-independent dimensionless quantity $(\epsilon_J\eta_J/\hbar)$ depends only on the space quantum number J, as seen from (27), the other significant dimensionless quantity $(\eta_J\hbar/\epsilon_J m^2c^2)$ may be an intrinsic property of the particle independent of J. Thus let

$$\frac{\eta_J\hbar}{\epsilon_J m^2c^2}=\theta, \qquad \forall J \tag{28}$$

where θ is independent of J.

(V) For the given values of m and L the quantities J, ϵ_J, and η_J are uniquely determined by the relations in (23), (27), and (28). Further J, ϵ_J, and η_J are such that

$$\text{when } L\to\infty, \qquad J\to\infty, \qquad \epsilon_J\to 0, \qquad 2J\epsilon_J\to\infty, \qquad \eta_J\to 0. \tag{29}$$

(VI) If an observable K of the particle is represented by the operator $\hat{K}(\hat{q},\hat{p})$ in the normal quantum mechanical formalism then it will now be represented by the matrix $K_J(Q_J,P_J)$ obtained from $\hat{K}(\hat{q},\hat{p})$ by the rule of replacement

$$\hat{q}\to Q_J, \qquad \hat{p}\to P_J \tag{30}$$

and the eigenvalues of the matrix $K_J(Q_J,P_J)$ are the values that the observable K can take. For example $\{P_J^2/2m+\frac{1}{2}m\omega^2Q_J^2\}$ will represent the Hamiltonian operator corresponding to the nonrelativistic harmonic oscillation of the particle with a frequency ω and the eigenvalues and the eigenvectors of the matrix $\{P_J^2/2m+\frac{1}{2}m\omega^2 Q_J^2\}$ will characterize the corresponding energy eigenstates.

(VII) Except for the replacement of the operators by finite-dimensional matrices all other aspects of the usual quantum theory can be assumed to be valid in general. For example, Born's probabilistic interpretation of state vectors is applicable, expectation values of the observables can be defined in the usual manner, Heisenberg's uncertainty principle exists since Q_J and P_J do not commute, time is regarded as an independent continuous parameter, and the temporal development of the state vector $|\psi(t)\rangle_J$ of the particle is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_J=H_J|\psi(t)\rangle_J \tag{31}$$

with H_J as the Hamiltonian matrix or for any observable K the rate of change of the corresponding matrix in the Heisenberg picture is defined by

$$\left(\frac{dK}{dt}\right)_J=\left(\frac{\partial K_J}{\partial t}\right)_t+\frac{i}{\hbar}[H_J, K_J] \tag{32}$$

We may call the quantum mechanical formalism based on the above postulates (I)–(VII) the "finite-dimensional quantum mechanics" (FDQM).

It is to be noted that while postulates (I)–(III) and (V)–(VII) are consequences of (5)–(21), (23), (27), and (28) and the general philosophy of the correspondence principle of Bohr, postulate (IV), based on the belief in the simplicity of the laws of Nature, is yet to be justified by a reasonably significant determination of the value of θ in (28). This will be achieved in the following sections.

Before closing this section let us observe the important fact that the above formulation of the FDQM conforms to the general philosophy of Bohr's correspondence principle, as mentioned above. Let a $(2J+1)$-dimensional vector $|\psi\rangle$ denote a quantum state of the above particle. For the sake of notational simplicity we shall omit hereafter the subscript J for ϵ_J, η_J, etc. whenever its value is clear from the context. Then following the Dirac notation we shall prescribe the components of $|\psi\rangle$ in position representation by $\{\langle n\epsilon|\psi\rangle | n=-J,\ldots, J\}$. Equations (24) and (25) imply that

$$\langle n\epsilon|Q\psi\rangle=n\epsilon\langle n\epsilon|\psi\rangle \tag{33}$$

$$\langle n\epsilon|P\psi\rangle=\frac{1}{(2J+1)}\sum_{n'=-J}^{J}\sum_{s=-J}^{J}s\eta\exp\left[\frac{i2\pi s(n-n')}{2J+1}\right]\langle n'\epsilon|\psi\rangle \tag{34}$$

When L is infinitely large J is also infinitely large and ϵ and η become infinitesimally small according to the postulate (V) above so that we can regard $n\epsilon$, $n'\epsilon$, and $s\eta$ in (33) and (34) as almost continuously varying from $-\infty$ to ∞. Then replacing the quasicontinuous variables $n\epsilon$, $n'\epsilon$, and $s\eta$, respectively by q, q', and p we can rewrite (33) and (34) with the aid of (27) as

$$\langle q|Q\psi\rangle = q\langle q|\psi\rangle = \langle q|\hat{q}\psi\rangle \tag{35}$$

$$\begin{aligned}
\langle q|p\psi\rangle &= \frac{1}{2J+1}\sum_{n'=-J}^{J}\sum_{s=-J}^{J} s\eta \exp\left[\frac{i2\pi s(n-n')}{2J+1}\right]\langle n'\epsilon|\psi\rangle \\
&= \frac{1}{(2J+1)\epsilon\eta}\sum_{n'\epsilon=-J\epsilon}^{J\epsilon}\epsilon\sum_{s\eta=-J\eta}^{J\eta}\eta s\eta\exp\left[\frac{i2\pi s\eta(n\epsilon-n'\epsilon)}{(2J+1)\eta\epsilon}\right]\langle n'\epsilon|\psi\rangle \\
&\approx \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dq'\int_{-\infty}^{\infty}dp\, p\exp\left[\frac{ip(q-q')}{\hbar}\right]\langle q'|\psi\rangle \\
&= -i\hbar\frac{d}{dq}\langle q|\psi\rangle \\
&= \langle q|\hat{p}\psi\rangle
\end{aligned} \tag{36}$$

Thus it follows that we may regard the application of the current quantum mechanical formalism with $\hat{q}$ and $\hat{p}$ equivalent to the Schrödinger representation

$$\langle q|\hat{q}\psi\rangle = q\langle q|\psi\rangle \tag{37}$$

$$\langle q|\hat{p}\psi\rangle = -i\hbar\frac{d}{dq}\langle q|\psi\rangle \tag{38}$$

in accordance with the Heisenberg relation (1) as really an asymptotic approximation valid in the case described by (I)–(VII) above only when L is infinitely large. In other words the FDQM characterized by a space quantum number J approaches the usual quantum mechanics irrespective of the mass of the particle when J becomes infinitely large as required by the general philosophy of Bohr's correspondence principle. But if L is small for a particle as for the constituents of atomic nuclei the FDQM based on the above postulates (I)–(VII) may be very appropriate.

3. ON THE VALUE OF θ

Let us first observe that the relations in (16)–(21) are completely analogous to the well-known relations in the usual quantum mechanics given by

$$\hat{S}\hat{q}\hat{S}^{-1}=\hat{p} \tag{39}$$

$$\hat{S}\hat{p}\hat{S}^{-1}=-\hat{q} \tag{40}$$

$$(\hat{S}^2)\hat{q}(\hat{S}^2)^{-1}=-\hat{q} \tag{41}$$

$$(\hat{S}^2)\hat{p}(\hat{S}^2)^{-1}=-\hat{p} \tag{42}$$

$$\langle q|\hat{S}^2|q'\rangle=\delta(q+q') \tag{43}$$

$$\langle q|\hat{S}^4|q'\rangle=\delta(q-q') \tag{44}$$

where $\hat{S}$ is the unitary Fourier transform operation defined by

$$\langle q|\hat{S}\psi\rangle=\frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty}dq'\exp\left(\frac{iqq'}{\hbar}\right)\langle q'|\psi\rangle \tag{45}$$

Actually in the asymptotic limit when $J\to\infty$ (16)–(21) must transform themselves into (39)–(44), respectively. Further it is evident that the choice of an odd number of quantized values for q and p in the FDQM is due to the requirement of existence of a parity operator in the form of S_J^2 as in (18)–(20) in exact analogy with the case of $\hat{S}^2$ which represents the parity operator in the usual quantum mechanics as seen from (41)–(43). We shall utilize this close analogy between the operator $\hat{S}$ and the matrix S_J for a determination of the important constant θ of the FDQM.

It is known that the operator $\hat{S}$ can be represented as

$$\hat{S}=\exp\left\{\frac{i\pi}{2}\left[\frac{1}{2\hbar}(\hat{p}^2+\hat{q}^2)-\frac{1}{2}\right]\right\}$$
$$=\exp\left[\frac{i\pi}{2}\left(\frac{\hat{\mathcal{H}}}{\hbar}-\frac{1}{2}\right)\right] \tag{46}$$

and hence the operator $[(i\pi/4\hbar)(\hat{p}^2+\hat{q}^2-\hbar)]$ may be regarded as the logarithm of $\hat{S}$ (see for instance Jauch, 1968; Wolf, 1979). Then it is seen that the operator $\hat{\mathcal{H}}$ occurring in (46) is the normal quantum mechanical

Hamiltonian operator corresponding to the linear nonrelativistic harmonic oscillation of a particle of unit mass with unit frequency. In analogy with this situation let us investigate whether the matrix S_J can be written in the form

$$S_J = \exp\left[\frac{i\pi}{2}\left(\frac{\mathcal{H}_J}{\hbar\omega_J} + \xi_J I_J\right)\right] \tag{47}$$

analogous to (46) where ξ_J is a constant and $\mathcal{H}_J$ is a Hamiltonian matrix associated with an "intrinsic" oscillatory motion of the particle of given mass m such that the pair of relations

$$\frac{i}{\hbar}[\mathcal{H}_J, Q_J] = \frac{P_J}{m} \tag{48}$$

$$\frac{i}{\hbar}[\mathcal{H}_J, P_J] = -m\omega_J^2 Q_J \tag{49}$$

is a result of the equations of motion given in (32) when it is required that in the Heisenberg picture

$$\left(\frac{dQ}{dt}\right)_J = \frac{P_J}{m} \tag{50}$$

$$\left(\frac{dP}{dt}\right)_J = -m\omega_J^2 Q_J \tag{51}$$

as is usual for a normal nonrelativistic quantum oscillator. The fact that in this case the Hamiltonian matrix $\mathcal{H}_J$ must be associated with an "intrinsic" harmonic oscillation of the particle is clear from the observation that the normal Hamiltonian matrix $\{P_J^2/2m + \frac{1}{2}m\omega_J^2 Q_J^2\}$ that would correspond to this case according to postulate (VI) above cannot satisfy (48) and (49) as required.

Now using (10), (11), and (15) we can rewrite (48) and (49) as

$$i[A_J, N_J] = \zeta_J \Phi_J \tag{52}$$

$$i[A_J, \Phi_J] = -(1/\zeta_J) N_J \tag{53}$$

respectively with

$$A_J = \frac{\mathcal{H}_J}{\hbar\omega_J} \tag{54}$$

$$\zeta_J = \frac{\eta_J}{m\omega_J \epsilon_J} \tag{55}$$

Then from (16), (17), (47), and (54) it is found that we must have

$$\exp\left(\frac{i\pi}{2}A_J\right)N_J\exp\left(-\frac{i\pi}{2}A_J\right)=\Phi_J \tag{56}$$

$$\exp\left(\frac{i\pi}{2}A_J\right)\Phi_J\exp\left(-\frac{i\pi}{2}A_J\right)=-N_J \tag{57}$$

But (52) and (53) would be consistent with (56) and (57) if and only if

$$\zeta_J=1 \tag{58}$$

or

$$\omega_J=\frac{\eta_J}{m\epsilon_J} \tag{59}$$

for any value of J. Hence (52) and (53) determining $A_J=(\mathcal{H}_J/\hbar\omega_J)$ reduce to

$$i[A_J, N_J]=\Phi_J \tag{60}$$

$$i[A_J, \Phi_J]=-N_J. \tag{61}$$

It can be now seen directly from (56), (57), (60), and (61) that $\{\exp[-(i\pi/2)A_J]\}S_J$ commutes with both N_J and Φ_J so that S_J and $\exp[(i\pi/2)A_J]$ must differ only by a constant multiple. Thus if (60) and (61) are consistent and lead to a solution for A_J then the logarithm of S_J is given by $(i\pi/2)A_J$ up to an additive constant and $\mathcal{H}_J=\hbar\omega_J A_J$ can be interpreted as the Hamiltonian operator for the "intrinsic" harmonic oscillation of the particle of rest mass m with a characteristic frequency $\omega_J=(\eta_J/m\epsilon_J)$.

Writing in terms of matrix elements and using (5) and (6), (60) and (61) are seen to give the relations

$$\langle n|A_J|n'\rangle=\frac{i\langle n|\Phi_J|n'\rangle}{n-n'}, \qquad n\neq n' \tag{62}$$

$$\langle n|A_J|n\rangle-\langle n'|A_J|n'\rangle$$
$$=\frac{i}{\langle n|\Phi_J|n'\rangle}\left\{n\delta_{nn'}-\sum_{\substack{r=-J\\(\neq n,n')}}^{J}\frac{\langle n|\Phi_J|r\rangle\langle r|\Phi_J|n'\rangle[2r-(n+n')]}{(n-r)(r-n')}\right\} \tag{63}$$

$$2\sum_{\substack{r=-J\\(\neq n)}}^{J}\frac{\langle r|\Phi_J|n\rangle^2}{(r-n)}=n, \qquad \forall n=0,\pm1,\ldots,\pm J \tag{64}$$

Among these relations (64) specifies the condition for (60) and (61) to be consistent leading to a solution for A_J. Using (6) it can be checked easily that for the case $J=1$, (64) is satisfied completely. But for any $J>1$ consideration of (64) for the case $n=1$, with the aid of (6), leads to absurd trigonometrical relations. This shows that only for the unique case, $J=1$, is the set of identities in (64) completely satisfied. Then (62) and (63) can be used to show easily that the matrix A_1 satisfying the relations in (60) and (61) for the case, $J=1$, is of the form

$$\begin{aligned} A_1(k) &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 2\sqrt{3}\,k+1 & -2 & 1 \\ -2 & 2\sqrt{3}\,k-2 & -2 \\ 1 & -2 & 2\sqrt{3}\,k+1 \end{pmatrix} \\ &= A_1(0)+kI_1 \end{aligned} \tag{65}$$

in general with any value for the number k. Since S_1 has eigenvalues $(1, i, -1)$ and $A_1(k)$ has eigenvalues $(k-1, k, k+1)$ it can be easily guessed that we must have

$$\log S_1 = (i\pi/2)A_1(1) \tag{66}$$

or we can write

$$S_1 = \exp[(i\pi/2)A_1(1)] \tag{67}$$

The truth of (67) can also be checked directly by computing $\exp[(i\pi/2)A_1(1)]$ through the process of diagonalization of $A_1(1)$. From the above discussion we know clearly that only in the case of the 3×3 matrix

$$\begin{aligned} S_1 &= \frac{1}{\sqrt{3}} \begin{pmatrix} w_3 & 1 & w_3^{-1} \\ 1 & 1 & 1 \\ w_3^{-1} & 1 & w_3 \end{pmatrix}, \\ w_3 &= \exp\left(\frac{2\pi i}{3}\right) \end{aligned} \tag{68}$$

which is equivalent to the 3×3 Sylvester matrix,

$$\mathbb{S}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & w_3^2 \\ 1 & w_3^2 & w_3 \end{pmatrix} \tag{69}$$

we can have a representation as

$$S_1 = \exp[(i\pi/2)A_1] \tag{70}$$

with

$$A_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2\sqrt{3}+1 & -2 & 1 \\ -2 & 2\sqrt{3}-2 & -2 \\ 1 & -2 & 2\sqrt{3}+1 \end{pmatrix} \tag{71}$$

where A_1 has the unique property that

$$i[A_1, N_1] = \Phi_1 \tag{72}$$

$$i[A_1, \Phi_1] = -N_1 \tag{73}$$

Thus it is very clear that only in the case $J=1$ are (47)–(49) consistent and meaningful so that there exists an intrinsic Hamiltonian matrix $\mathcal{H}_1$ of oscillator type associated with the particle of rest mass m such that

$$\frac{i}{\hbar}[\mathcal{H}_1, Q_1] = \frac{P_1}{m} \tag{74}$$

$$\frac{i}{\hbar}[\mathcal{H}_1, P_1] = -m\omega_1^2 Q_1 \tag{75}$$

$$\omega_1 = \frac{\eta_1}{m\epsilon_1} \tag{76}$$

$$S_1 = \exp\left[\frac{i\pi}{2}\left(\frac{\mathcal{H}_1}{\hbar\omega_1} + \xi_1 I_1\right)\right] \tag{77}$$

Motivated by the fact that the above unique Hamiltonian matrix $\mathcal{H}_1$ must be of great physical significance we shall choose that

$$\mathcal{H}_1 = \hbar\omega_1 A_1(0) \tag{78}$$

$$\xi_1 = 1 \tag{79}$$

so that in view of (65) we have

$$S_1 = \exp\left[\frac{i\pi}{2}\left(\frac{\mathcal{H}_1}{\hbar\omega_1} + I_1\right)\right]$$
$$= \exp\left[\frac{i\pi}{2} A_1(1)\right] \tag{80}$$

consistent with (67).

Since $A_1(0)$ has eigenvalues $(0, \pm 1)$ as already seen $\mathcal{H}_1$ has eigenvalues $(0, \pm\hbar\omega_1)$ as implied by (78). The unique case, $J=1$, corresponds to the situation where the particle is confined to a region of the least possible dimension $L_1 = 2\epsilon_1$. So it is very natural to consider that the energy eigenvalues of the corresponding unique intrinsic Hamiltonian $\mathcal{H}_1$ must be associated with the intrinsic energy states of the particle. Then because of the fact that Planck's quantum of energy, $\hbar\omega_0$, associated with the de Broglie frequency of the particle at rest, $\omega_0 = mc^2/\hbar$, would be the rest energy of the particle, $E_0 = mc^2$, as given by Einstein, we shall assume that

$$\omega_1 \equiv \omega_0 = mc^2/\hbar \tag{81}$$

so that the energy eigenvalues of the intrinsic Hamiltonian $\mathcal{H}_1$ become 0, mc^2, and $-mc^2$ corresponding to the vacuum or no-particle, particle, and antiparticle states, respectively. It is very interesting to observe that the above discussion naturally brings out a close connection between Planck's quantum hypothesis, Einstein's mass–energy relation, de Broglie's wave theory of matter, and the concept of antiparticle, the product of Dirac's relativistic wave equation. Equations (76) and (81) imply that

$$\frac{\eta_1 \hbar}{\epsilon_1 m^2 c^2} = 1 \tag{82}$$

Now in view of our postulate (IV), in general, we must have

$$\frac{\eta_J \hbar}{\epsilon_J m^2 c^2} = \theta = 1, \qquad \forall J \tag{83}$$

Let us close this section with the following important remarks:

(a) Relation (83) points out that even in the asymptotic limit $J \to \infty$ or the case of continuous q spectrum we should expect the quantity (η/ϵ) to have the value $(m^2c^2/\hbar)$ when $\epsilon \to 0$ and $\eta \to 0$ according to the postulate (V). But at first sight a comparison of (16) and (17) with (39) and (40) seems to imply that in this case $(\eta/\epsilon) = 1$. This apparent contradiction disappears if we consider the fact that the quantity (η/ϵ) may have any nonzero value in the asymptotic limit when the spectra of q and p are continuous in the range $[-\infty, \infty]$.

(b) In our opinion the above one-dimensional model of a mechanism with an intrinsic Hamiltonian $\mathcal{H}_1$ associated with some internal property of the particle responsible for the constant value of (η_J/ϵ_J) should not be regarded as a purely one-dimensional "mechanical" model which should be suitably generalized in the three-dimensional case and rather we should regard the result in (83) to be true in case of any direction in three-dimensional space irrespective of the situation whether the particle has freedom of motion in the other directions or not. Thus while generalizing the one-dimensional form of the FDQM given here to the three-dimensional case the relation in (83) must be assumed to be valid in all the directions independent of each other.

4. DETERMINATION OF J, ϵ_J, AND η_J FOR A PARTICLE WITH GIVEN VALUE FOR L

From (23), (27), (28), and (83) it is seen that for a particle of rest mass m the basic relations among the quantities L, J, ϵ_J, and η_J which decide the physical behavior of the particle are given by

$$\epsilon_J \eta_J = \frac{2\pi\hbar}{2J+1} \tag{84}$$

$$\frac{\eta_J}{\epsilon_J} = \frac{m^2 c^2}{\hbar} \tag{85}$$

$$J = \text{integer} > 0 \quad \text{such that } 2J\epsilon_J \leq L < 2(J+1)\epsilon_{J+1}. \tag{86}$$

Equations (84) and (85) show that for any particular value of J, ϵ_J, and η_J are given uniquely by

$$\epsilon_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} \frac{\lambda_c}{2\pi}, \qquad \lambda_c = \frac{2\pi\hbar}{mc} \tag{87}$$

$$\eta_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} mc \tag{88}$$

where λ_c is the well-known Compton wavelength of the particle. Then the "space quantum number" J for a given value of L is fixed uniquely by the condition

$$\left[\frac{2J^2}{(2J+1)\pi}\right]^{1/2} \leqslant \frac{L}{\lambda_c} < \left[\frac{2(J+1)^2}{(2J+3)\pi}\right]^{1/2} \tag{89}$$

as implied by (86) and (87). It is easy to verify that (87)–(89) satisfy the condition in (29) as required.

Let us now calculate the velocity spectrum of the particle corresponding to the momentum spectrum given by (26) and (88) using the formula

$$p = \frac{mv}{\left[1-(v^2/c^2)\right]^{1/2}} \tag{90}$$

due to Einstein. Then it is seen that the velocity spectrum given by

$$v_{Jn} = n\left(\frac{2\pi}{2\pi n^2 + 2J + 1}\right)^{1/2} c, \qquad n = -J, \ldots, -1, 0, 1, \ldots, J \tag{91}$$

is independent of the rest mass of the particle and is characterized purely by the space quantum number J. From (91) it is seen that for a given value of J the magnitude of the maximum allowed value of v, say V_J, is given by

$$V_J = v_{JJ} = \left(\frac{2\pi J^2}{2\pi J^2 + 2J + 1}\right)^{1/2} c \tag{92}$$

and is such that $\lim_{J\to\infty} V_J \to c$ as it should be. This fact has an interesting implication as follows. If a photon has a nonvanishing rest mass (see for instance, Jauch and Rohrlich, 1955, for a discussion of this possibility), however small it may be, then it would be reasonable only if we associate it, in the above picture, with the case, $J = 1$ and $L_1 \approx$ diameter of the universe. But then paradoxically the magnitude of the maximum velocity of the photon, namely, V_1, would have the minimum allowed nonzero value for any particle. Thus the consistency of the picture with relativistic concepts requires that photons must have zero rest mass.

5. CONCLUSION

Apart from the requirement that a photon must have zero rest mass let us point out also the following interesting examples of the implications of the above theory. Since the lowest value of J is 1, $L_1 = 2\epsilon_1 = (8\pi/3)^{1/2}(\hbar/mc)$ should be the minimum dimension of a region which can hold a particle of rest mass m. Thus for an electron with $m_e \approx 9\times 10^{-31}$ kg we find $L_{1e} \approx 10^{-12}M = 10^3$ fm. This again confirms the well-known fact that an electron cannot reside inside an atomic nucleus. But for a nucleon with $m_n \approx 1838 m_e$ we find that $L_{1n} \approx 0.61$ fm, which is of the right order of magnitude for the existence of nucleons within nuclei. Further it is worth noticing that this value of L_{1n} is very close to the present estimate of the radius of repulsive hard core of the nucleon–nucleon force (see for instance, Cohen, 1971).

Thus from the above considerations we can conclude as follows. Let us consider that the application of current quantum mechanical formalism with the representation of basic position operator $\hat{q}$ and momentum operator $\hat{p}$ equivalent to the Schrödinger representation

$$\langle q|\hat{q}\psi\rangle = q\langle q|\psi\rangle \tag{93}$$

$$\langle q|\hat{p}\psi\rangle = -i\hbar\frac{d}{dq}\langle q|\psi\rangle \tag{94}$$

in accordance with the Heisenberg relation

$$[\hat{q}, \hat{p}] = i\hbar \tag{95}$$

as really an asymptotic approximation to the actual situation where position q and momentum p of a particle of rest mass m moving within a finite one-dimensional region of fixed length L take only certain quantized values as given by

$$q = n\epsilon_J, \qquad n = 0, \pm 1, \pm 2, \ldots, \pm J \tag{96}$$

$$p = n\eta_J, \qquad n = 0, \pm 1, \pm 2, \ldots, \pm J \tag{97}$$

$$J = \text{integer} > 0 \quad \text{such that } 2J\epsilon_J \leqslant L < 2(J+1)\epsilon_{J+1} \tag{98}$$

Then for a particle of given rest mass m with a fixed value for L the basic quantities J, ϵ_J, and η_J are determined by the quantization rules:

$$\epsilon_J = \left(\frac{2\pi}{2J+1}\right)^{1/2}\frac{\lambda_c}{2\pi}, \qquad \lambda_c = \frac{2\pi\hbar}{mc} \tag{99}$$

$$\eta_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} mc \tag{100}$$

$$\left[\frac{2J^2}{(2J+1)\pi}\right]^{1/2} \leqslant \frac{L}{\lambda_c} < \left[\frac{2(J+1)^2}{(2J+3)\pi}\right]^{1/2} \tag{101}$$

It seems quite plausible that the quantum mechanical formalism applicable for particles confined to move within very small regions of space, such as for the constituents of atomic nuclei, takes a finite-dimensional form corresponding to small values of the "space quantum number" J. In that case the basic position operator Q and the conjugate momentum operator P will have to be represented by $(2J+1)$-dimensional matrices given, respectively, by

$$\langle n|Q_J|n'\rangle = n\epsilon_J\delta_{nn'} \tag{102}$$

$$\langle n|P_J|n'\rangle = \begin{cases} 0 & \text{if } n=n' \\ \dfrac{i\eta_J}{2}\csc\left[\dfrac{2\pi J(n-n')}{2J+1}\right] & \text{if } n\neq n' \end{cases}$$

$$n, n' = -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \tag{103}$$

The other operators of the particle must be obtained from their normal quantum counterparts by the replacement procedure

$$\hat{q}\to Q_J, \qquad \hat{p}\to P_J \tag{104}$$

Except for the replacement of the Schrödinger operators by finite-dimensional matrices according to the rules in (99)–(104) all other aspects of the usual quantum theory are valid in general.

The extension of the above scheme to the isotropic three-dimensional case should be straightforward with only a few minor necessary modifications if we assume that the quantization rules in (99)–(101) are valid for any direction and independent coordinate operators commute with each other as usual in the normal quantum theory.

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